Math 142 Lecture 24 Notes

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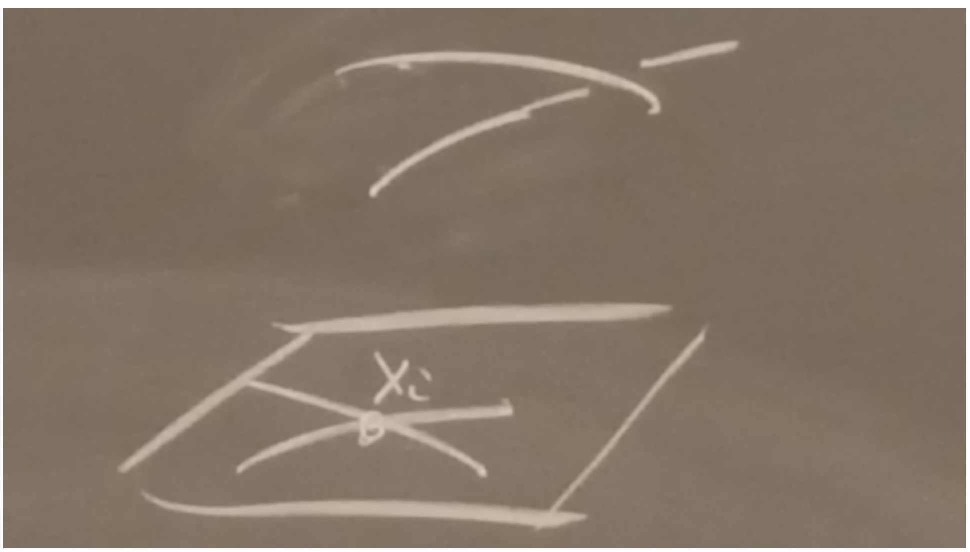
1 Knot Colorings

1.1 Tricolorings

Let's start off by giving a rigorous definition of arc.

Definition 1.1. Given a nice projection, each double point $x_i \in \mathbb{R}^2$ has 2 preimages in K: (x, y, z_1) and (x, y, z_2), where $z_1 < z_2$.



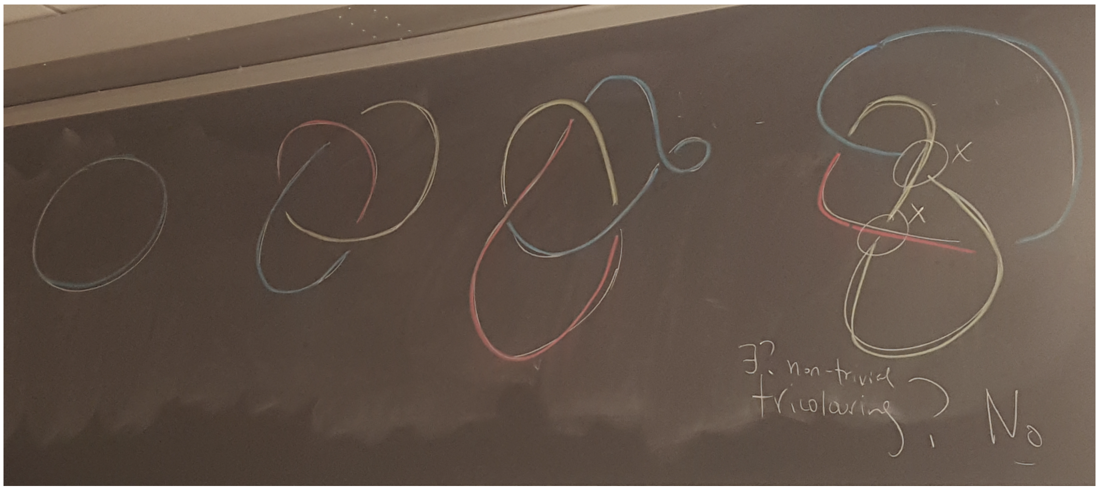
Take a neighborhood $A_i \subseteq K$ for (x, y, z) such that $A_i \cong (0, 1)$ (and is small). Then an *arc* is a connected component of $K \setminus (A_1 \cup \cdots \cup A_n)$.

Now that we have this definition, we won't use it ever again.[1]

We defined arcs to talk about tricoloring. Recall that a knot is tricolorable if there exists a nice projection of the knot with a non-trivial tricoloring.

[1] I personally lament the sentiment behind this.

Example 1.1. Let's try to make tricolorings of the unknot, trefoil knot, a modified (by R1) trefoil knot, and a figure-eight knot. If we start with the added loop on the modified trefoil, we must have only 1 color on that crossing (since there are only two arcs involved in the crossing, we cannot have 3 colors). If we start with one arc in the figure-eight knot and color it blue, we can see that there is no nontrivial tricoloring.

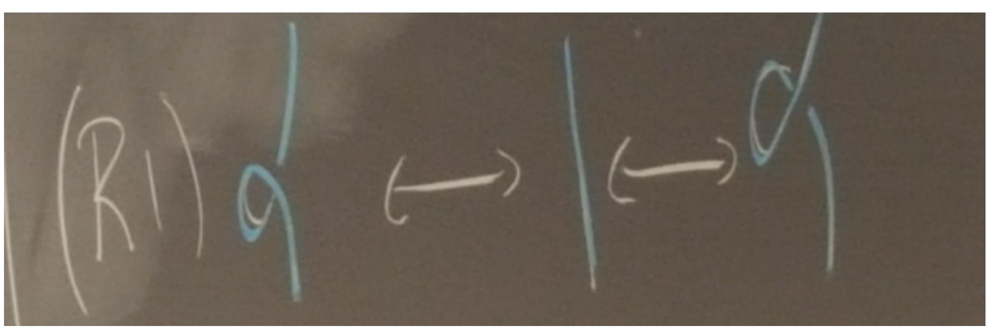


How do we know that we didn't just choose a bad projection of the figure-eight knot? What if another projection is tricolorable?

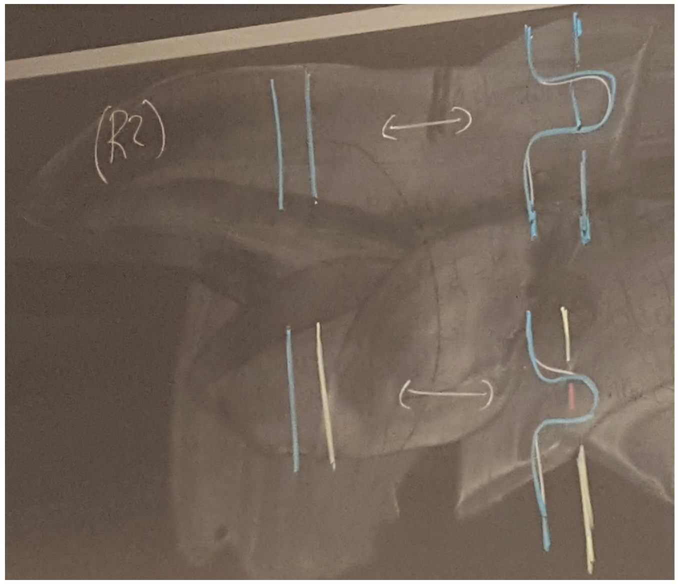
Proposition 1.1. *If a knot is tricolorable, then any nice projection of it (and any knot isotopic to it) has a nontrivial tricoloring.*

Proof. We need to check that the existence of nontrivial tricolorings is independent of projection. So we show it doesn't change under Reidemeister moves.

1. (R1): If we have a self loop crossing, it can only have 1 color because there are only two arcs involved. So

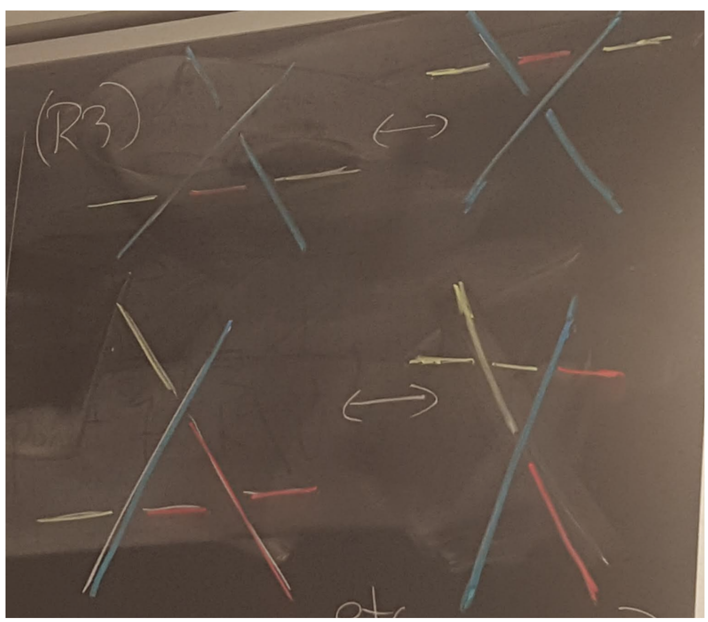


2. (R2): There are two main cases; we leave the rest as an exercise.



In each case, there is only one choice for the tricoloring of the modified picture; this means that there is a bijection between the tricolorings.

3. (R3) The ends have to be the same color before and after the Reidemeister move so we can "patch in" this section into the knot. This gives us one possible coloring in each case given a coloring before doing the Reidemeister move.



□

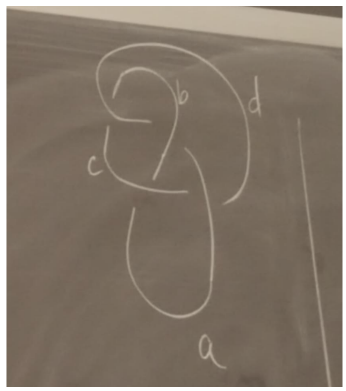
This proof actually shows the following.

Corollary 1.1. *The number of tricolorings of a projection of a knot K is independent of the projection.*

We also get the following corollary.

Corollary 1.2. *The trefoils are not isotopic to the unknot or to the figure-eight knot.*

Proof. The trefoil is tricolorable. The unknot has a projection with no nontrivial colorings, so the proposition implies that the unknot is not tricolorable. The figure-eight knot has a projection



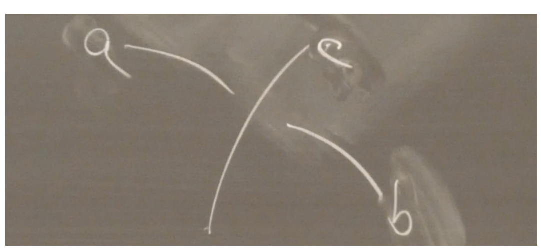
with $a, b, c, d \in \{1, 2, 3\}$. If $a = b$, then one of the crossings makes $a = c$. Then another crossing gives $a = d$, so we get the trivial coloring. If $a \neq b$, say $a = 1$ and $b = 2$, then the first crossing gives us $c = 3$. Another crossing then gives us $d = 2$. But a third crossing contains b, c, and d, so we have a crossing with 2, 3, and 2, which is impossible. Since this projection of the figure-eight knot has no nontrivial tricoloring, the proposition implies that the figure-eight knot is not tricolorable. $\square$

This result leaves us with a question: is the unknot isotopic to the figure-eight knot?

1.2 n-colorings

Let's generalize the idea of 3-colorings.

Definition 1.2. An *n-coloring* of a nice projection is a choice of color $1, \ldots, n$ for each arc such that at each crossing



we have $2c \equiv a + b \pmod{n}$. A *trivial n-coloring* is a one with only one color. A knot is *n-colorable* if there exists a nontrivial n-coloring of a nice projection of a knot isotopic to K.

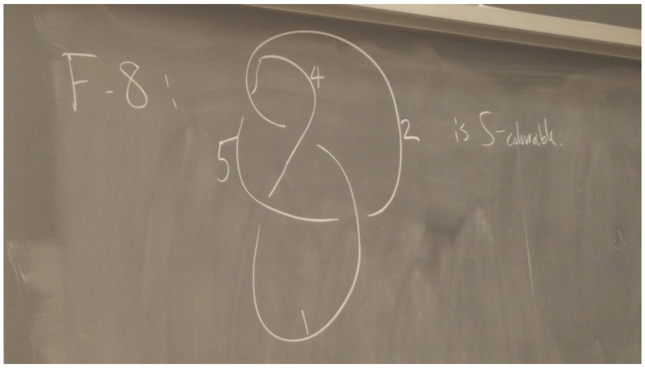
For $n = 3$, $2c - b - a \equiv 0 \pmod{3}$ iff $a = b = c$ or $\{a, b, c\} = \{1, 2, 3\}$. Check this yourself.

Theorem 1.1. *K is n-colorable iff any nice projection of K has a nontrivial n-coloring. The number of n-colorings of a nice projection of K is independent of the projection.*

Proof. The proof is the same as the 3-coloring case. Check (R1)-(R3). $\square$

Corollary 1.3. *The figure-eight knot is not isotopic to the unknot.*

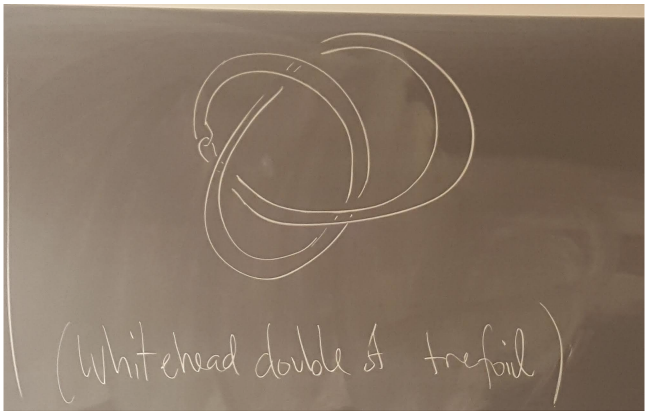
Proof. The unknot is not 5-colorable. The figure-eight knot, however, is 5-colorable. Check that the following 5-coloring works.



Note that we don't have to use all 5 of the colors to get a nontrivial 5-coloring. $\square$

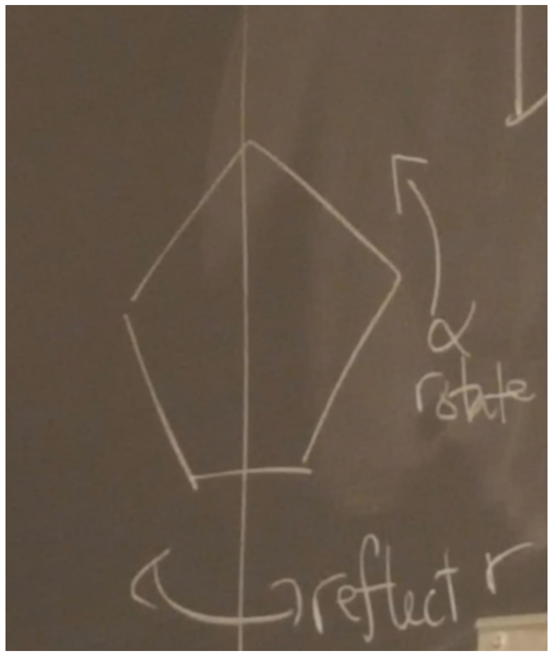
Remark 1.1. There exist nontrivial knots that are not n-colorable for any n.

Example 1.2. Here is a nontrivial knot that is not n-colorable for any n. It is called the *Whitehead double* of the trefoil.



Our goal for the next few lectures is to relate n-colorings to algebraic topology. We will show that the set of n-colorings is almost in bijection with the set of homomorphisms $\pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$, where D_{2n} is the dihedral group of symmetries of the n-gon.

$$D_{2n} = \left\langle r, \alpha \mid r^2 = 1, \alpha^n - 1, \alpha r = r\alpha^{-1} \right\rangle.$$



We will calculate $\pi_1(\mathbb{R}^3 \setminus K)$ from a projection. It will have 1 generator x_i for every arc. We will look at homomorphisms $\phi : \pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$ sending $x_i \mapsto r\alpha^{c_i}$, where c_i is the color for arc i.